

# THE SPIN PARAMETER OF UNIFORMLY ROTATING COMPACT STARS

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## ABSTRACT

We study the dimensionless spin parameter  $j (= cJ/(GM^2))$  of uniformly rotating neutron stars and quark stars in general relativity. We show numerically that the maximum value of the spin parameter of a neutron star rotating at the Keplerian frequency is  $j_{\max} \sim 0.7$  for a wide class of realistic equations of state. This upper bound is insensitive to the mass of the neutron star if the mass of the star is larger than about  $1 M_{\odot}$ . On the other hand, the spin parameter of a quark star modeled by the MIT bag model can be larger than unity and does not have a universal upper bound. Its value also depends strongly on the bag constant and the mass of the star. Astrophysical implications of our finding will be discussed.

*Subject headings:* Dense matter—stars: neutron—stars: rotation

## 1. INTRODUCTION

The general stationary vacuum solution (the Kerr spacetime) of the Einstein equations is specified uniquely by the gravitational mass  $M$  and the angular momentum  $J$  (see, e.g., Wald (1984)). If  $J \leq GM^2/c$ , we have a rotating black hole. However, if  $J > GM^2/c$ , the Kerr spacetime would have a naked singularity without a horizon. One could then consider closed timelike curves and causality would be violated (Chandrasekhar 1983). While its validity has not yet been proven, the cosmic-censorship conjecture (Penrose 1969) asserts that naked singularities cannot be formed via the gravitational collapse of a body. For this reason, it is believed that astrophysical black holes should satisfy the Kerr bound  $j \leq 1$ , where  $j = cJ/GM^2$  is the dimensionless spin parameter.

While the value of the spin parameter  $j$  plays a fundamental role in black-hole physics, it appears that this is not the case for other stellar objects. In particular, there is no theoretical constraint on the value of  $j$  for stars. It is known that the spin parameter of main-sequence stars depends sensitively on the stellar mass and can be much larger than unity (Kraft 1968, 1970; Dicke 1970; Gray 1982). On the other hand, the spin parameter of compact stars has not been studied in detail (see below). As we shall discuss in more detail in Section 3, the spin parameter of compact stars is interesting in its own right for two reasons. (1) It plays a role in our understanding of the observed quasi-periodic oscillations (QPOs) in disk-accreting compact-star systems. (2) It determines the final fate of the collapse of a rotating compact star.

Ever since the seminal work of Hartle (1967) who considered the limit of slow rotation, rotating compact stars have been studied extensively in general relativity. In the past two decades, various different numerical codes have been developed to construct rapidly rotating stellar models in general relativity. We refer the reader to Stergioulas (2003) for a review. As the rotation frequency  $f$  is a directly measurable quantity for pulsars, it is thus reasonable that the max-

imum value for  $f$  (i.e., the Keplerian frequency  $f_K$ ) has been one of the most studied physical quantities for relativistic rotating stars (see, e.g., Cook et al. (1994); Haensel et al. (1995); Koranda et al. (1997); Benhar et al. (2005); Haensel et al. (2009)). However, in contrast to these previous works, we shall focus extensively on the spin parameter  $j$ .

It has been known that the spin parameter for a maximum-mass neutron star lies in the range  $j \sim 0.6 - 0.7$  for most realistic equations of state (EOS; e.g., Cook et al. (1994) and Salgado et al. (1994)). In this work, we first extend the previous works and show that the upper bound for the spin parameter  $j_{\max} \sim 0.7$  is essentially independent of the mass of the neutron star if the gravitational mass of the star is larger than  $\sim 1 M_{\odot}$ . Next we study the spin parameter of self-bound quark stars, which was not considered previously in Cook et al. (1994) and Salgado et al. (1994). We find that the behavior of the spin parameters of neutron stars and quark stars is very different. In contrast to the case of neutron stars, the spin parameter of quark stars does not have a universal upper bound. It also depends sensitively on the parameter of the quark matter EOS and the mass of the star. Furthermore, the spin parameter of quark stars can be larger than unity. This leads us to propose that the spin parameter could be a useful indicator to identify rapidly rotating quark stars.

The plan of this paper is as follows. Section 2 presents the main numerical results of this work. In Section 3, we discuss the astrophysical implications of our results. Our conclusions are summarized in Section 4.

## 2. NUMERICAL RESULTS

### 2.1. Numerical method and EOS models

We make use of the numerical code **rotstar** from the C++ LORENE library<sup>1</sup> to calculate uniformly rotating compact star models in general relativity. The code uses a multi-domain spectral method (Bonazzola et al. 1998) to solve the Einstein equations in a stationary and axisymmetric spacetime with a matter source

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<sup>1</sup> <http://www.lorene.obspm.fr/>

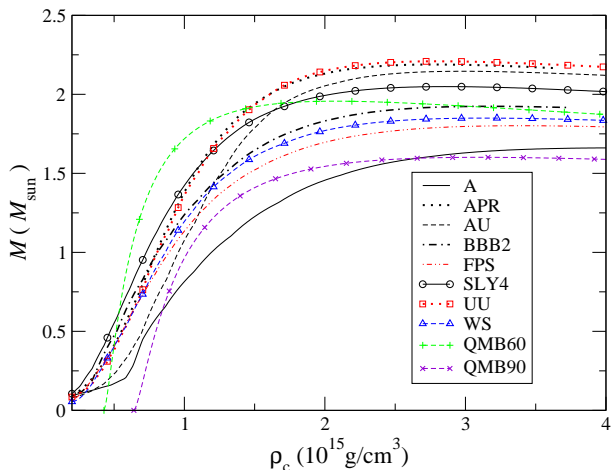


FIG. 1.— Gravitational mass as a function of the central energy density for non-rotating compact stars constructed with the chosen EOS models in this work.

(Bonazzola et al. 1993; Gourgoulhon et al. 1999). The code has been tested extensively and compared with a few different numerical codes (Nozawa et al. 1998).

As theoretical calculations for dense matter at supranuclear densities are poorly constrained, the EOS in the high-density core of compact stars is not well understood (see, e.g., Weber et al. (2007); Haensel et al. (2007) for reviews). In this work, we employ eight realistic nuclear matter EOS to model rotating neutron stars: model A (Pandharipande 1971), model APR (Akmal et al. 1998), model AU (the AV14+UVII model in Wiringa et al. (1988) is joined to Negele & Vautherin (1973)), model BBB2 (Baldo et al. 1997), model FPS (Pandharipande & Ravenhall 1989; Lorenz et al. 1993), model SLY4 (Douchin & Haensel 2000), model UU (the UV14+UVII model in Wiringa et al. (1988) is joined to Negele & Vautherin (1973)) and model WS (the UV14+TNI model in Wiringa et al. (1988) is joined to Lorenz et al. (1993)). For the quark star models, we use the simplest MIT bag model with non-interacting massless quarks (Chodos et al. 1974). Two different values,  $60 \text{ MeV fm}^{-3}$  and  $90 \text{ MeV fm}^{-3}$ , are chosen for the bag constant  $B$ . These values correspond approximately to the range of  $B$  within which the hypothesis of strange matter is valid (Haensel et al. 2007).

To illustrate the diversity of the EOS models used in this work, we plot the gravitational mass  $M$  against central energy density  $\rho_c$  for non-rotating stars constructed with the chosen EOS models in Figure 1. The quark star models QMB60 and QMB90 in the figure correspond to the cases  $B = 60 \text{ MeV fm}^{-3}$  and  $90 \text{ MeV fm}^{-3}$ , respectively. The maximum mass of compact stars depends quite sensitively on the EOS models and it ranges from about  $1.5 M_\odot$  to  $2.2 M_\odot$ . In Figure 2, we plot the Keplerian frequency  $f_K$  against the gravitational mass  $M$  of rotating compact stars based on our chosen EOS models. Similar to the maximum mass for non-rotating compact stars, Figure 2 shows clearly that  $f_K$  depends strongly on the EOS model. It is also sensitive to the mass of the star. This is the well-known reason why searching for rapidly rotating compact stars can provide us constraints on the EOS of dense matter.

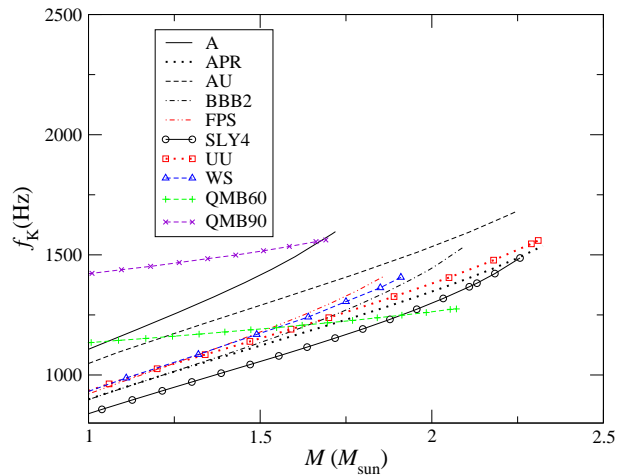


FIG. 2.— Keplerian frequency as a function of the gravitational mass for rotating compact stars constructed with the chosen EOS models in this work.

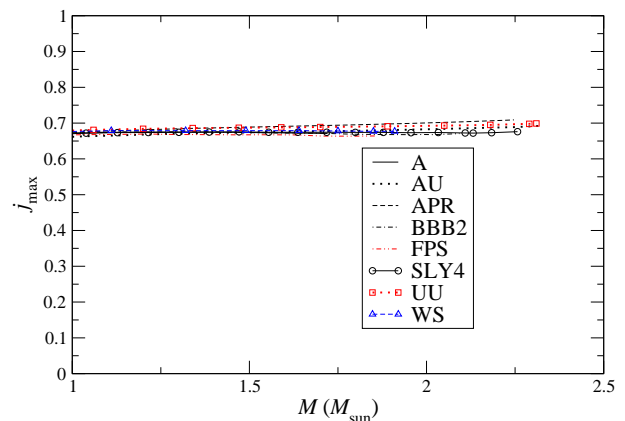


FIG. 3.— Maximum spin parameter as a function of the gravitational mass for rotating neutron stars.

## 2.2. Neutron stars

Now we turn to the main focus of this work: the dimensionless spin parameter  $j$ . Having seen that  $f_K$  depends strongly on the EOS and the mass of the star, it may be quite surprising to learn that the maximum value of the spin parameter  $j_{\text{max}}$  (as set by the Kepler limit) is quite universal for rotating neutron stars. In Figure 3, we plot  $j_{\text{max}}$  against the gravitational mass  $M$  for the selected nuclear matter EOS models. In the figure, each line corresponds to one particular EOS and each point on a line corresponds to a star model with a fixed  $M$  rotating at its Keplerian frequency. Note that each sequence in the figure is terminated at the stellar model with the same total particle number as the stable maximum-mass non-rotating configuration. In the figure, we see that  $j_{\text{max}}$  lies in a narrow range  $\sim 0.65 - 0.7$  for the eight different nuclear matter EOS models. In particular, the values do not depend sensitively on the mass of the star. This extends previous works (Cook et al. 1994; Salgado et al. 1994) which focus on maximum-mass neutron star models.

While the spin parameter of an astrophysical black hole is constrained by  $j \leq 1$ , we find that the spin parameter

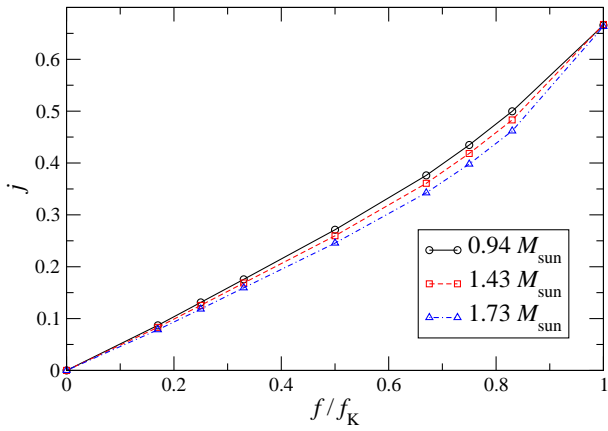


FIG. 4.— Spin parameter is plotted against the scaled rotation frequency for neutron stars constructed with the FPS EOS. Each line represents a sequence of fixed total particle number. Each sequence is labeled by the gravitational mass of its non-rotating configuration.

of a neutron star is bounded by  $j_{\max} \sim 0.7$ . The upper bound  $j_{\max}$  is quite universal for different EOS models and gravitational mass larger than  $\sim 1 M_{\odot}$ . For lower mass neutron stars,  $M < 1 M_{\odot}$ , we find that  $j$  decreases with decreasing  $M$ . However, we shall only focus on mass  $M > 1 M_{\odot}$  in this work as the observed masses of neutron stars are typically larger than  $1 M_{\odot}$ . We refer the reader to Steiner et al. (2010) for a recent review on the observed masses of neutron stars.

So far we have studied the maximum spin parameter  $j_{\max}$  of neutron stars rotating at their Keplerian frequencies  $f_K$ . However, realistic neutron stars in general rotate slower with frequencies  $f < f_K$ . Is the spin parameter  $j$  still insensitive to the EOS and mass of the star? In Figure 4, we plot the spin parameter  $j$  against the scaled rotation frequency  $f/f_K$  for the FPS EOS. In the figure, each line represents a sequence of fixed total particle number (the so-called evolutionary sequence). Each sequence is labeled by the gravitational mass of its non-rotating configuration  $M_0$ . The solid line corresponds to  $M_0 = 0.94 M_{\odot}$ , the dashed line corresponds to  $M_0 = 1.43 M_{\odot}$ , and the dashed-dotted line corresponds to  $M_0 = 1.73 M_{\odot}$ . Note, however, that the gravitational mass increases with rotation frequency. The figure shows that the spin parameter changes only by at most 10% (depending on  $f/f_K$ ) when  $M_0$  changes from 0.94 to  $1.73 M_{\odot}$ . It is also interesting to note that the differences between the curves decrease as the scaled frequency  $f/f_K$  tends to 1. This is quite different from the case of quark stars which will be studied shortly. In Figure 5, we plot  $j$  against  $f/f_K$  for three different EOS models. The stellar models on the three sequences have the same total particle numbers such that their baryonic masses are fixed at  $M_B = 1.6 M_{\odot}$ . It is seen that, for a fixed scaled frequency, the spin parameter of neutron stars is essentially independent of the EOS models.

### 2.3. Quark stars

Let us now consider self-bound quark stars using the MIT bag model. In Figure 6, we plot  $j_{\max}$  against  $M$  for the two quark matter models QMB60 and QMB90. In contrast to the case of neutron stars (see Figure 3), we see that  $j_{\max}$  depends sensitively on the mass of the star.

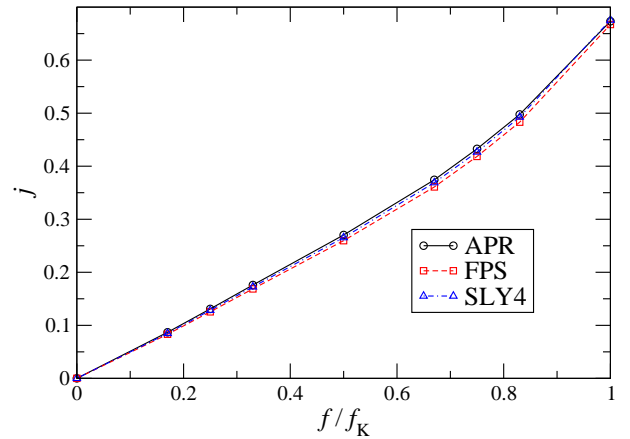


FIG. 5.— Spin parameter is plotted against the scaled rotation frequency for neutron stars constructed with three different EOS models. The baryonic mass is fixed at  $M_B = 1.6 M_{\odot}$ .

For the QMB60 model,  $j_{\max}$  is decreased by about 24% as  $M$  increases from  $1 M_{\odot}$  to  $2 M_{\odot}$ . Comparing to the case of neutron stars, it is also seen that  $j_{\max}$  has a more significant dependence on the EOS parameter, namely the bag constant in the MIT bag model. Analogous to Figure 4 for neutron stars, we plot  $j$  against  $f/f_K$  for the QMB60 model in Figure 7. Each line represents a sequence of fixed total particle number. As in Figure 4, each sequence is labeled by the gravitational mass of its non-rotating configuration  $M_0$ . We show three different sequences in the figure:  $M_0 = 0.82 M_{\odot}$  (solid),  $1.27 M_{\odot}$  (dashed), and  $1.55 M_{\odot}$  (dash-dotted). We see that the differences between the curves increase significantly as the scaled frequency increases. At  $f/f_K = 1$ , the spin parameter is increased by about 27% when  $M_0$  changes from  $1.55$  to  $0.82 M_{\odot}$ .

We see that the spin parameter of quark stars can be significantly larger than the upper bound ( $j_{\max} \sim 0.7$ ) for neutron stars. It can also be larger than the Kerr bound  $j = 1$  for black holes. This suggests that the spin parameter could be a useful indicator to identify rapidly rotating quark stars. Discovering even one single compact star with spin parameter  $j \gtrsim 0.7$  will provide a strong evidence for the existence of quark stars. Finally, it should be pointed out that the fact that the spin parameter of quark stars can be larger than 0.7 is also evident from the results of Stergioulas et al. (1999). Using the values of the gravitational mass  $M$  and angular momentum  $J$  for Keplerian quark stars presented in Table 1 of Stergioulas et al. (1999), it is easy to check that the quark stars considered by the authors all have  $j_{\max} > 0.7$ . In particular, their results also suggest that  $j_{\max}$  decreases with increasing  $M$  as we have seen in Figure 6.

## 3. ASTROPHYSICAL IMPLICATIONS

We have studied the spin parameter of uniformly rotating compact stars in general relativity. Our numerical results show that the behavior of the spin parameter of quark stars is quite different from that of neutron stars. In particular, the spin parameter of neutron stars is bounded above by  $j_{\max} \approx 0.7$ , while quark stars can have a value larger than unity. In this section, we shall discuss (in our view) the astrophysical implications of

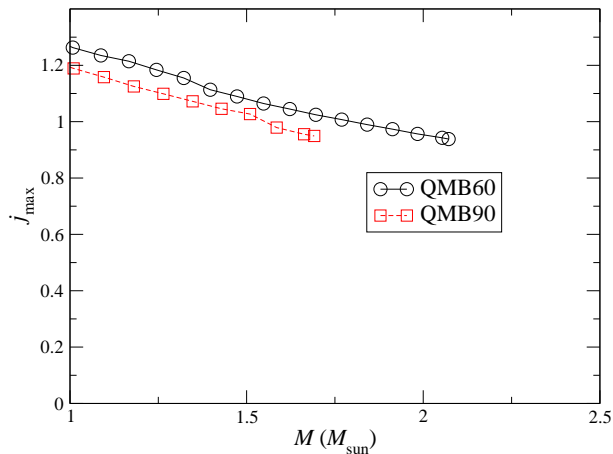


FIG. 6.— Maximum spin parameter as a function of the gravitational mass for rotating quark stars.

our results.

First, how could the spin parameter of a compact star be measured? Unfortunately, so far there is no general technique to infer the spin parameter  $j$  of compact stars directly. As far as we are aware, the spin parameter of a compact star could be potentially measured in disk-accreting compact-star systems. In particular, the neutron stars (or quark stars) in low-mass X-ray binaries (LMXBs) provide the most natural cosmic laboratories for studying the spin parameter. In order to understand how the spin parameter might be inferred in disk-accreting systems, it should be noted that the spin parameter of the central compact star directly affects the particle motion around the star. For example, to first order in  $j$ , the orbital frequency of a point particle in a prograde orbit around a compact star is given by (see, e.g., van der Klis (2006))

$$2\pi\nu_\phi = \left[ 1 - j \left( \frac{GM}{rc^2} \right)^{3/2} \right] \left( \frac{GM}{r^3} \right)^{1/2}, \quad (1)$$

where  $r$  is the orbital radius. For infinitesimally tilted and eccentric orbits, the disk particles will have radial ( $\nu_r$ ) and vertical ( $\nu_\theta$ ) epicyclic frequencies which also depend on  $j$  (see van der Klis (2006) for the expressions). Furthermore, the combination  $\nu_\theta - \nu_r$  also gives rise to the periastron frequency of the orbit. These frequencies in general depend on  $M$  and  $j$ , and hence their observations (possibly needed to be combined with the measurement of other stellar parameters such as the mass) would in principle provide useful information on the spin parameter. In fact, there are strong evidences that these frequencies have already been observed in LMXBs.

It should be noted that existing algebraic relations, such as Equation (1), which relate various orbital frequencies to stellar parameters are in general only valid for small spin rate  $j \sim 0.1$ . The difficulty of obtaining the corresponding algebraic relations for rapidly rotating stars ( $j \sim 1$ ) lies in the fact that there is no exact analytic representation of the vacuum spacetime outside a rapidly rotating compact star. Having such an analytic representation of the spacetime metric will allow one to obtain the desired algebraic relations for a rapidly rotating compact star. A starting point along this direction would be to take the closed-form asymptotically

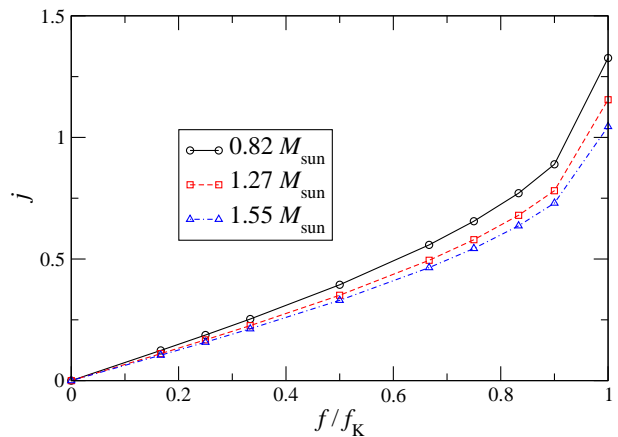


FIG. 7.— Spin parameter is plotted against the scaled rotation frequency for quark stars constructed with the QMB60 EOS. Each line represents a sequence of fixed total particle number. Each sequence is labeled by the gravitational mass of its non-rotating configuration.

flat solution of the Einstein-Maxwell system obtained by Manko et al. (2000) and study the geodesics in this spacetime. If there is no charge and magnetic moment, this analytic solution depends only on the mass, angular momentum, and quadrupole moment of the spacetime. Furthermore, this solution only involves rational functions, which helps to simplify the analytical study of geodesic motions. Berti & Stergioulas (2004) have demonstrated that this analytic solution can describe the exterior spacetime of a rapidly rotating neutron star (e.g.,  $j > 0.5$ ) very well. Nevertheless, further investigation is needed to check whether this analytic solution is also valid for rapidly rotating quark stars.

One of the well-observed features of LMXBs is the high-frequency ( $\sim$  kHz) QPOs. To date, QPOs have been observed in more than 20 LMXBs. These QPOs often come in pairs with frequencies  $\nu_u$  and  $\nu_l$ . In all systems in which the spin frequencies of the compact stars  $\nu_{\text{star}}$  have been measured, the frequency separation  $\Delta\nu = \nu_u - \nu_l$  is approximately equal to  $\nu_{\text{star}}$  or  $\nu_{\text{star}}/2$ . We refer the reader to van der Klis (2006) and Lamb & Boutloukos (2008) for recent reviews. While the physical mechanism responsible for producing the high-frequency QPOs is not known yet, most physical models involve orbital motion and disk oscillations. Hence, the frequencies  $\nu_r$ ,  $\nu_\theta$ ,  $\nu_\phi$  and various combinations of them are often invoked (either directly or indirectly) to explain high-frequency QPOs. This is the reason why one might hope to obtain useful information on the spin parameter of the central compact star in an LMXB by observing its QPOs. For example, if the higher frequency of the QPO pair ( $\nu_u$ ) is identified with the orbital frequency ( $\nu_\phi$ ), one can then obtain a relation between the spin parameter and an upper bound on the mass of the compact star (Miller et al. 1998). On the other hand, based on the so-called relativistic precession model of QPOs (Stella & Vietri 1998, 1999), Török et al. (2010) have recently derived a constraint relating the mass and spin parameter of the compact star in Cir X-1. These works thus demonstrate the possibility of measuring the value of  $j$  with an independent determination of the mass and vice versa.

The above brief review of high-frequency QPOs serves to point out the astrophysical relevance of the spin parameter and in what situations could it be potentially measured. Now we are ready to discuss how one could make use of the finding in this work to obtain useful information regarding the central compact star in LMXBs. Suppose a single well established model for the high-frequency QPOs can be agreed upon in the future, then it is possible that rapidly rotating quark stars could be identified from the inferred spin parameters. As shown in our numerical results, if the inferred spin parameter of the central star is larger than  $\sim 0.7$ , then the star could be a quark star. Of course, the star can either be a neutron star or quark star if the spin parameter is less than 0.7.

On the other hand, for the present situation where many models are available, our finding could still be used to put constraints on the physical models for QPOs. For example, let us consider the resonantly excited disk-oscillation model for QPOs proposed by Kato (2008). In the model, Kato suggests that the high-frequency QPOs are inertial-acoustic oscillations on a deformed disk that are resonantly excited by nonlinear couplings between the oscillation modes and the disk deformation. Kato applies the model to Cir X-1 and finds that it describes the observed QPOs quite well if the mass of the central star is about  $1.5 - 2.0 M_{\odot}$  and the spin parameter is  $j \sim 0.8$ . However, our work suggests that uniformly rotating neutron stars cannot have  $j \gtrsim 0.7$ . If Kato's model is correct, then our finding implies that the central star in Cir X-1 could be a quark star. On the other hand, if other measurements in the future (such as the mass-radius relation or cooling property of the central star) suggest that the compact star in the system is indeed a traditional neutron star, then our finding would rule out the resonantly excited disk-oscillation model.

After discussing how one might use the spin parameter to distinguish between neutron stars and quark stars in LMXBs, let us now turn to a different issue concerning the collapse of a rotating star to black hole. In the past decade, general relativistic simulations of rotational collapse of neutron stars modeled by the polytropic EOS have been performed (Shibata et al. 2000a; Shibata 2003a,b; Duez et al. 2004; Baiotti et al. 2005a). These simulations show that no stable massive disks can be formed around the resulting black holes. The implication is that the collapse of a uniformly rotating neutron star (with  $j < 1$ ) to black hole cannot lead to the black-hole accretion model of gamma-ray burst, which requires a rotating black hole surrounded by an accretion disk (see Piran (2005) for a review). It is expected that the initial spin parameter of the collapsing star must be  $j \gtrsim 1$  in order to form a massive disk around the final black hole (Shibata 2003a,b; Duez et al. 2004). As rapidly rotating quark stars can have  $j > 1$ , it is thus possible that the collapse of a rapidly rotating quark star could be a progenitor for the black-hole accretion model of gamma-ray burst. Furthermore, it is noted that the collapse of massive stars to black holes (the so-called collapsars) could form massive disks around the black holes and hence produce long gamma-ray bursts. Since quark stars have smaller mass ( $\sim M_{\odot}$ ) comparing to massive stars, the collapse of quark stars might only produce small disks. As the duration of the accretion, and hence the timescale

of the bursts, depends on the mass of the disk, the collapse of quark stars might thus be a mechanism for short gamma-ray bursts.

On the other hand, what if the collapse process does not lead to any (or little) mass ejection as in the collapse of neutron stars? In fact, Bauswein et al. (2009) have recently simulated quark star mergers based on the MIT bag model and the conformally flat approximation to general relativity. Their results show that the mass ejection depends sensitively on the bag constant  $B$ . In particular, they find that there are some binary mergers without mass ejection. Could this conclusion, namely the absence of mass ejection for some values of  $B$ , also be true for the rotational collapse of quark stars? For the collapse of rotating neutron stars to black holes, apart from the small amount of total mass energy  $M$  and angular momentum  $J$  carried away by gravitational radiation (Baiotti et al. 2005b), the final black holes have essentially the same  $M$  and  $J$ , and hence the same spin parameter  $j$ , as the initial star. For a rapidly rotating quark star with initial spin parameter  $j > 1$ , if there is no mass ejection, how could the spin parameter be reduced efficiently in order to form a regular black hole that satisfies the Kerr bound  $j \leq 1$  at the end of the collapse? If a Kerr black hole could not be formed in the process, then what would be the final fate of the collapse? These questions deserve further investigation using fully general relativistic modeling. The hope is that studying the collapse of quark stars might lead to the discovery of some new phenomena which are not seen in the collapse of neutron stars. This might then help to shed more light on the highly nonlinear dynamics of gravitational collapse in general relativity.

The astrophysical implications discussed above depend crucially on the existence of rapidly rotating quark stars with  $j > 0.7$ . However, it is known that rapidly rotating compact stars may be subject to different kinds of secular or dynamical non-axisymmetric mode instabilities (Andersson 2003). One might thus worry that rotating quark stars (if they exist) may be limited to small spin rates due to various instabilities, and hence rendering a possible distinction between neutron stars and quark stars unlikely to happen. However, Gondek-Rosińska et al. (2003) have shown that, taking into account realistic values of shear viscosity, viscosity-driven bar mode instability cannot develop in quark stars modeled by the MIT bag model in any astrophysically relevant temperature windows. Even taking the unrealistic assumption of infinite shear viscosity, the instability can develop only if the ratio of the rotational kinetic energy to the absolute value of the gravitational potential energy  $T/W$  is larger than 0.1375. The exact value depends on the stellar mass. For comparison, a quark star modeled by the MIT bag model, with  $B = 60 \text{ MeV fm}^{-3}$ ,  $M = 1.146 M_{\odot}$ , and  $j = 0.8$ , has the value  $T/W = 0.126$ . We have followed the same numerical procedure of Gondek-Rosińska et al. (2003) to check that this quark star is indeed stable against the viscosity-driven instability. We refer the reader to Gondek-Rosińska & Gourgoulhon (2002) and Gondek-Rosińska et al. (2003) for more details on the numerical procedure. On the other hand, Gondek-Rosińska et al. (2003) have also discussed that

the gravitational-radiation driven  $r$ -mode instability seems to be unimportant for quark stars in LMXBs if the strange quark mass is  $m_s c^2 \sim 200$  MeV (standard value) or higher. The  $r$ -mode instability can develop and may limit quark stars to small spin rates only if the strange quark mass takes the relatively low value  $m_s c^2 \sim 100$  MeV.

Even if the viscosity-driven and gravitational-radiation-driven instabilities (which are both secular effects) cannot develop in a quark star, the star may still be subject to dynamical bar-mode instability which occurs at a higher spin rate. This kind of instability has not been studied for rotating quark stars. However, we can still obtain some insight from the Newtonian theory of a rotating incompressible star, which is a good approximation to quark stars because of their rather uniform density profile. The onset of dynamical instability occurs at  $T/W \approx 0.27$  for an incompressible star (Chandrasekhar 1969). For comparison, a quark star modeled by the MIT bag model, with  $B = 60$  MeV fm $^{-3}$ ,  $M = 1.4 M_\odot$  and  $j = 1.11$ , has the value  $T/W = 0.23$ . Although general relativity may change the critical value  $T/W$  for the onset of the instability to a somewhat smaller value than that is suggested by the Newtonian theory<sup>2</sup>, it is still very likely that rapidly rotating quark stars with  $j > 0.7$  can exist. Some of them may even break the Kerr bound  $j = 1$  for black holes.

#### 4. CONCLUSIONS

In this paper, we have studied the dimensionless spin parameter  $j$  of uniformly rotating neutron stars and quark stars in general relativity. We find that the maximum value of the spin parameter (as set by the Kepler limit) of neutron stars is bounded above by  $j_{\max} \sim 0.7$ . This upper bound is essentially independent of the EOS of the neutron star. It is also insensitive to the mass

of the star if the mass of the star is larger than about  $1 M_\odot$ . On the other hand, the spin parameter of quark stars behaves quite differently. We find that the spin parameter of quark stars modeled by the MIT bag model can be larger than unity. It also depends sensitively on the EOS parameter (i.e., the bag constant) and the mass of the star.

We have discussed (in our view) the astrophysical implications of our finding in detail in Section 3. We have discussed how the spin parameter of compact stars could be potentially measured in LMXBs and its relevance in the physical models for high-frequency QPOs. As a first application, our finding implies that the compact star in Cir X-1 could be a quark star if the resonantly excited disk-oscillation model for QPOs is correct (Kato 2008), since the model requires that the spin parameter of the central star to be  $j \sim 0.8$  in order to fit the observed QPOs. We have also speculated on how the collapse of a rotating quark star might be different from the collapse of a neutron star. As explained in Section 3, the collapse of a rapidly rotating quark star with  $j \gtrsim 1$  might lead to the formation of a stable disk around the resulting Kerr black hole. This implies that the collapse process might form the central engine of a gamma-ray burst. However, a fully general relativistic dynamical calculation (which has not been done in this work) is required to shed light on the issue.

In conclusion, our work suggests that discovering even one single compact star with spin parameter  $j \gtrsim 0.7$  will provide a strong evidence for the existence of quark stars, and hence verifying the hypothesis that strange quark matter could be absolutely stable (Witten 1984).

#### ACKNOWLEDGMENTS

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